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Supersymmetry on a euclidean spacetime lattice 1. A target theory with four supercharges

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Abstract: We formulate a euclidean spacetime lattice whose continuum limit is $(2,2)$ supersymmetric Yang-Mills theory in two dimensions, a theory which possesses four supercharges and an anomalous global chiral symmetry. The lattice action respects one exact supersymmetry, which allows the target theory to emerge in the continuum limit without fine-tuning. Our method exploits an orbifold construction described previously for spatial lattices in Minkowski space, and can be generalized to more complicated theories with additional supersymmetry and more spacetime dimensions.

Keywords: Lattice Gauge Field Theories, Extended Supersymmetry.

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Contents

1. Introduction and results

Supersymmetric Yang-Mills (SYM) theories are extraordinarily rich, exhibiting a variety of complex nonperturbative phenomena. Much is known analytically about such theories, yet presumably much more could be learned from numerical study. Lattice versions of supersymmetric field theories have been discussed at length in the literature [1]–[20] (for a recent review see [21]). A persistent obstacle to numerical simulation of supersymmetric theories is the lack of continuous translational invariance, which makes exact realization of the desired supersymmetry on the lattice impossible. Without this exact supersymmetry, fine-tuning seems the only approach to obtaining the supersymmetric continuum limit. The problem is especially acute for the most interesting case of non-minimal SYM theories, which contain spin zero interacting fields: without exact lattice supersymmetry it would seem that fine-tuning of the bare lattice parameters would be needed to eliminate radiative corrections to scalar masses and interactions.

A method to circumvent this problem was recently proposed for spatial lattices with continuous Minkowski time [22]. The lattices are obtained by an orbifold projection of a "mother theory" that has as much supersymmetry as the target theory. The projection leaves a subset of the supersymmetries realized exactly on the lattice, protecting the theory from unwanted relevant operators in the continuum limit.[1] It was shown that the desired continuum theory, with its enhanced super and chiral symmetries, can be obtained without fine tuning. This procedure, inspired by recent work on orbifolding [23]–[26] and the deconstruction of supersymmetric theories [27]–[30], leads to rather unorthodox lattices, with spin zero bosons and single component fermions living on both links and sites, as well as noncompact gauge fields. The lattice structures also tend to be more exotic than the d-cubic lattices normally considered in d dimensions.[2]

This is the first of several papers in which we extend the previous work to euclidean spacetime lattices, which are more suitable for numerical simulation. Here we work through a simple example in some detail: the lattice for $(2,2)$ SYM theory in two spacetime dimensions with $\mathrm{U}(k)$ gauge symmetry.[3] This target theory is described by the lagrangian

$$\mathcal{L} = \frac{1}{g_2^2}\,\mathrm{Tr}\left(\left|D_m s\right|^2 + \overline{\psi}\, iD_m\gamma_m\psi + \frac{1}{4}v_{mn}v_{mn} + i\sqrt{2}(\overline{\psi}_L[s,\psi_R] + \overline{\psi}_R[s^\dagger,\psi_L]) + \frac{1}{2}[s^\dagger, s\,]^2\right), \tag{1.1}$$

where ψ_R and ψ_L are the right- and left-chiral components of a two-component Dirac field ψ, s is a complex scalar field, v_m is the two dimensional gauge potential, $D_m = \partial_m + i[v_m,\,\cdot\,]$ is the covariant derivative, and $v_{mn} = -i[D_m, D_n]$ is the field strength. All fields are rank-k matrices transforming as the adjoint representation of $\mathrm{U}(k)$. This target theory has four supercharges and a chiral $\mathrm{U}(1)$ R-symmetry, and can be obtained by dimensionally reducing the more familiar $\mathcal{N}=1$ SYM theory in four dimensions down to two dimensions.

A naïve latticization of this model breaks both the supersymmetry and the chiral $\mathrm{U}(1)$ symmetry, requiring fine tuning to eliminate radiatively induced operators which violate these symmetries. Here we construct a supersymmetric lattice action which does not suffer from these problems, exhibiting both supersymmetry and chiral symmetry in the continuum without the need for fine-tuning of parameters. Our resulting lattice action is

$$\begin{aligned} S = \frac{1}{g^2}\sum_{\mathbf{n}}\mathrm{Tr}\Bigg[&\frac{1}{2}\left(\overline{x}_{\mathbf{n}-\hat{\mathbf{i}}}x_{\mathbf{n}-\hat{\mathbf{i}}} - x_{\mathbf{n}}\overline{x}_{\mathbf{n}} + \overline{y}_{\mathbf{n}-\hat{\mathbf{j}}}y_{\mathbf{n}-\hat{\mathbf{j}}} - y_{\mathbf{n}}\overline{y}_{\mathbf{n}}\right)^2 + 2\left|x_{\mathbf{n}}y_{\mathbf{n}+\hat{\mathbf{i}}} - y_{\mathbf{n}}x_{\mathbf{n}+\hat{\mathbf{j}}}\right|^2 + \\ &+ \sqrt{2}\left(\alpha_{\mathbf{n}}\overline{x}_{\mathbf{n}}\lambda_{\mathbf{n}} - \alpha_{\mathbf{n}-\hat{\mathbf{i}}}\lambda_{\mathbf{n}}\overline{x}_{\mathbf{n}-\hat{\mathbf{i}}}\right) + \sqrt{2}\left(\beta_{\mathbf{n}}\overline{y}_{\mathbf{n}}\lambda_{\mathbf{n}} - \beta_{\mathbf{n}-\hat{\mathbf{j}}}\lambda_{\mathbf{n}}\overline{y}_{\mathbf{n}-\hat{\mathbf{j}}}\right) - \\ &- \sqrt{2}\left(\alpha_{\mathbf{n}}y_{\mathbf{n}+\hat{\mathbf{i}}}\xi_{\mathbf{n}} - \alpha_{\mathbf{n}+\hat{\mathbf{j}}}\xi_{\mathbf{n}}y_{\mathbf{n}}\right) + \sqrt{2}\left(\beta_{\mathbf{n}}x_{\mathbf{n}+\hat{\mathbf{j}}}\xi_{\mathbf{n}} - \beta_{\mathbf{n}+\hat{\mathbf{i}}}\xi_{\mathbf{n}}x_{\mathbf{n}}\right) + \\ &+ a^2\mu^2\left(x_{\mathbf{n}}\overline{x}_{\mathbf{n}} - \frac{1}{2a^2}\right)^2 + a^2\mu^2\left(y_{\mathbf{n}}\overline{y}_{\mathbf{n}} - \frac{1}{2a^2}\right)^2\Bigg]. \end{aligned} \tag{1.2}$$

Here the vectors $\mathbf{n} \equiv \{n_x, n_y\}$ label the sites of the 2-dimensional $N\times N$ periodic lattice pictured in figure 1. All variables are $k\times k$ matrices; Latin variables are bosonic, while Greek variables are Grassmann. The variables $x, \overline{x}$ and α reside on the links in the $\hat{\mathbf{i}} = \{1,0\}$ direction, $y, \overline{y}$ and β are variables on the links in the $\hat{\mathbf{j}} = \{0,1\}$ direction, ξ sits on the diagonal links, and λ is a site variable. The parameter a appearing in the action has

[1] For other recent work on maintaining exact supersymmetries on the lattice, see [15]–[18].

[2] For related work on nonsupersymmetric theories, see [31]–[33].

[3] This model has been studied numerically using Discrete Light-Cone Quantization techniques in ref. [34]

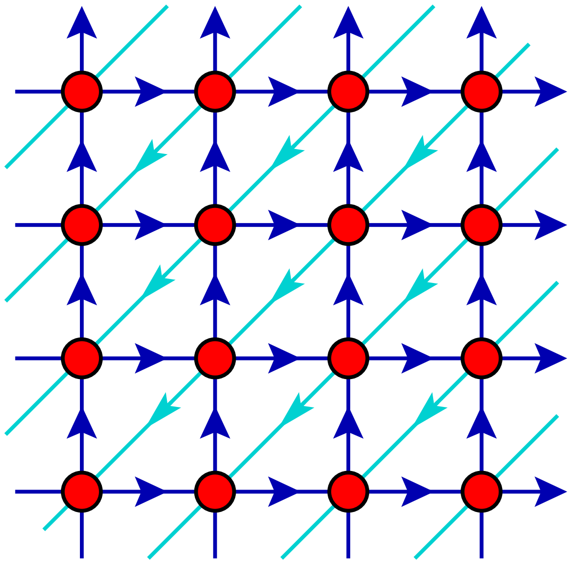

Figure 1: The two dimensional euclidean lattice for the two dimensional SYM theory with four supercharges. The lattice respects one exact supercharge. Arrows on the links signify the orientation of the $\mathbf{r}$ charge vectors of the link fermions, as discussed in §2.

dimension of length, and is interpreted as the lattice spacing. The continuum limit is defined as $a \to 0, N \to \infty$ while holding the 2-dimensional coupling $g_2 \equiv ga$ and the lattice size $L \equiv Na$ fixed.

The term proportional to μ^2 softly breaks the exact supersymmetry, and in the continuum limit μ is scaled such that $g_2 a \ll \mu L \lesssim 1$. The soft supersymmetry breaking effects then vanish in the large volume limit.

The remainder of this paper examines the route which leads to the peculiar lattice action eq. (1.2). The lattice arises from an orbifold projection of a matrix model with four exact supercharges. We examine the symmetries of the resulting lattice action and show that for $\mu = 0$ it preserves one exact supersymmetry. We demonstrate that the continuum limit of the lattice is the SYM theory of eq. (1.1), and that the exact lattice supersymmetry obviates the need for fine tuning in the quantum theory. As an aside, note that the lattice eq. (1.2) is an example of an interacting theory for which the critical point for obtaining chiral symmetry in the continuum limit is exactly known.

2. Constructing the lattice action

Our technique for obtaining a euclidean spacetime lattice begins with an $\mathcal{N} = 1$ SYM theory with $\mathrm{U}(kN^2)$ gauge group and $\mathcal{Q} = 4$ supercharges in $d = 4$ (euclidean) dimensions.[4] This theory is then dimensionally reduced to zero dimensions, yielding a matrix model containing both bosonic and fermionic rank-kN^2 matrices transforming as $\Phi \to g\Phi g^{-1}$

[4]Euclidean supersymmetry has been discussed in detail in ref. [35].

under the symmetry $\mathcal{G} = \mathrm{U}(kN^2)$. The action of this "mother theory" is given by

$$S = \frac{1}{g^2}\left(\frac{1}{4}\,\mathrm{Tr}\, v_{mn}v_{mn} + \mathrm{Tr}\,\overline{\psi}\,\overline{\sigma}_m[v_m, \psi]\right), \tag{2.1}$$

where $m, n = 0, \ldots, 3$, ψ and $\overline{\psi}$ are independent complex two-component spinors, v_m is the 4-vector of gauge potentials, and

$$v_{mn} = i[v_m, v_n]\,, \qquad \sigma_m = \{1,\, -i\boldsymbol{\sigma}\}\,, \qquad \overline{\sigma}_m = \{1,\, i\boldsymbol{\sigma}\}\,. \tag{2.2}$$

The "daughter theory" we will derive from this by orbifolding will be a two-dimensional lattice with N^2 sites and a $\mathrm{U}(k)$ symmetry associated with each site.

This mother action possesses an R-symmetry $G_R = \mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. The $\mathrm{SU}(2)\times\mathrm{SU}(2)\simeq\mathrm{SO}(4)$ subgroup of G_R is the Lorentz symmetry of the euclidean theory prior to dimensional reduction, under which

$$v \equiv v_m\overline{\sigma}_m \to LvR^\dagger\,, \qquad \overline{\psi}\to\overline{\psi}L^\dagger\,, \qquad \psi \to R\psi\,, \tag{2.3}$$

where L and R are independent $\mathrm{SU}(2)$ matrices. The $\mathrm{U}(1)$ factor of G_R is just the chiral symmetry of the gauginos under which

$$\psi \to e^{i\theta/2}\psi\,, \qquad \overline{\psi}\to\overline{\psi}e^{-i\theta/2}\,, \qquad v_m \to v_m\,. \tag{2.4}$$

We will denote the seven generators of the group $G_R = \mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ by

$$L_a\,,\; R_a\,,\; Y\,, \tag{2.5}$$

where L_a and R_a are $\mathrm{SU}(2)$ generators and the $\mathrm{U}(1)$ generator Y is normalized so that ψ and $\overline{\psi}$ carry charges $Y = \pm\frac{1}{2}$.

In addition to this R-symmetry the mother theory eq. (2.1) is also invariant under four independent supersymmetry transformations parametrized by two-component Grassmann spinors κ and $\overline{\kappa}$:

$$\delta v_m = -i\overline{\psi}\,\overline{\sigma}_m\kappa + i\overline{\kappa}\,\overline{\sigma}_m\psi\,, \qquad \delta\psi = -iv_{mn}\sigma_{mn}\kappa\,, \qquad \delta\overline{\psi} = iv_{mn}\,\overline{\kappa}\,\overline{\sigma}_{mn}\,, \tag{2.6}$$

where

$$\sigma_{mn} \equiv \frac{i}{4}\left(\sigma_m\overline{\sigma}_n - \sigma_n\overline{\sigma}_m\right), \qquad \overline{\sigma}_{mn} \equiv \frac{i}{4}\left(\overline{\sigma}_m\sigma_n - \overline{\sigma}_n\sigma_m\right). \tag{2.7}$$

A two dimensional lattice with N^2 sites may be created out of the rank-kN^2 matrices of the mother theory, following the procedure in [22]. We define a $\mathbb{Z}_N\times\mathbb{Z}_N\equiv\Gamma$ subgroup of $\mathcal{G}\times G_R$ and project out all field components not invariant under Γ. The embedding of Γ is designed to produce a lattice with nearest and next-to-nearest neighbor interactions, and one exact supersymmetry. The generators of the two $\mathbb{Z}_N$ factors in Γ are denoted as $\hat{\gamma}_1$, $\hat{\gamma}_2$, and their action on a field Φ of the mother theory is

$$\hat{\gamma}_a\Phi = e^{2\pi i r_a/N}\mathcal{C}^{(a)}\Phi{\mathcal{C}^{(a)}}^{-1} \qquad \text{with} \quad a = 1, 2 \tag{2.8}$$

where $e^{2\pi i r_a/N}\in G_R$ and $\mathcal{C}^{(a)}\in\mathcal{G}$. The $\mathcal{C}^{(a)}$ are referred to as "clock" matrices. It is convenient to write our rank-kN^2 matrix field Φ in the form $\Phi_{\mu\nu i,\,\mu'\nu' j}$ with Greek letters running from 1 to N, and Latin letters running from 1 to k. The clock matrix $\mathcal{C}^{(1)}$ in eq. (2.8) is taken to act nontrivially on the μ, μ' indices, while $\mathcal{C}^{(2)}$ acts on ν, ν'. In this notation, the $\mathcal{C}^{(a)}$ are the $\mathcal{G} = \mathrm{U}(kN^2)$ matrices

$$\mathcal{C}^{(1)} = \Omega \otimes \mathbf{1}_N \otimes \mathbf{1}_k\,, \qquad \mathcal{C}^{(2)} = \mathbf{1}_N \otimes \Omega \otimes \mathbf{1}_k\,, \tag{2.9}$$

where $\mathbf{1}_d$ signifies a rank-d unit matrix, and Ω is the rank-N unitary matrix

$$\Omega = \begin{pmatrix} \omega & & & \\ & \omega^2 & & \\ & & \ddots & \\ & & & \omega^N \end{pmatrix} \qquad \text{with} \quad \omega \equiv e^{2\pi i/N}\,. \tag{2.10}$$

The integer charges $\mathbf{r} = \{r_1, r_2\}$ in eq. (2.8) are constructed from the Cartan sub-algebra of $G_R = \mathrm{SU}_L(2) \times \mathrm{SU}_R(2) \times \mathrm{U}_Y(1)$:

$$r_1 = -L_3 + R_3 - Y\,, \qquad r_2 = +L_3 + R_3 - Y\,. \tag{2.11}$$

The orbifold projection eliminates the components of the mother theory fields which are not invariant under the Γ transformation defined in eq. (2.8). The projection operator may be written as

$$\hat{P} = \frac{1}{N^2} \sum_{m,n=1}^{N} (\hat{\gamma}_1)^m (\hat{\gamma}_2)^n\,. \tag{2.12}$$

The "daughter" theory is obtained by replacing every field Φ in the action of the mother theory by its projection $\tilde{\Phi} = \hat{P}\Phi$. Each projected field $\tilde{\Phi}$ consists of N^2 nonzero $k \times k$ blocks; the position of these nonzero blocks within the original rank-$k\,N^2$ matrix is determined by the integer values of the $r_{1,2}$ charges of Φ. Choosing a basis for the mother theory fields such that the r_a charges are diagonal, the nonzero blocks in $\tilde{\Phi}_{\mu\nu i,\mu'\nu' j}$ occur only for

$$\mu' = \mu + r_1\,, \qquad \nu' = \nu + r_2\,, \tag{2.13}$$

where r_1 and r_2 refer to the particular charges carried by that field Φ. Each nonzero rank-k block in $\tilde{\Phi}$ is labeled by the quartet of indices $\{\mu, \nu\}, \{\mu', \nu'\}$. We can consider these blocks as lattice variables of a two dimensional lattice, with each lattice site labeled by the pair $\{\mu, \nu\}$. Each block is then a link variable, residing on the link between sites $\{\mu, \nu\}$ and $\{\mu', \nu'\}$ — or residing at site $\{\mu, \nu\}$ if $\mu = \mu'$ and $\nu = \nu'$. We see from eq. (2.13) that fields with $\mathbf{r} = \{0, 0\}$ are site variables, while those with non-zero $\mathbf{r}$ are variables for links pointing in the $\mathbf{r}$ direction.

The variables of the mother theory are easily expressed in terms of eigenstates of the $\mathbf{r}$ charges:

$$x = \frac{v_0 - iv_3}{\sqrt{2}}\,, \qquad \overline{x} = \frac{v_0 + iv_3}{\sqrt{2}}\,, \qquad y = -i\frac{v_1 - iv_2}{\sqrt{2}}\,, \qquad \overline{y} = i\frac{v_1 + iv_2}{\sqrt{2}}\,. \tag{2.14}$$

and

$$\psi = \begin{pmatrix} \lambda \\ \xi \end{pmatrix}, \qquad \overline{\psi} = \begin{pmatrix} \alpha & \beta \end{pmatrix}\,. \tag{2.15}$$

	x	$\overline{x}$	y	$\overline{y}$	λ	ξ	α	β
L_3	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	$-\frac{1}{2}$	$+\frac{1}{2}$
R_3	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0
Y	0	0	0	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
r_1	$+1$	-1	0	0	0	-1	$+1$	0
r_2	0	0	$+1$	-1	0	-1	0	$+1$

Table 1: The $r_{1,2}$ charges of the fields of the $\mathcal{Q} = 4$ mother theory which define the orbifold projection.

The charge assignments for each of the variables in this action are shown in table 1. They follow from the transformations given in eq. (2.3) and eq. (2.4), along with the definition of the r_a in eq. (2.11). The daughter theory lattice we obtain after orbifold projection is shown in figure 1. In terms of these rank-kN^2 matrices, the action of the mother theory eq. (2.1) may be written as

$$S = \frac{1}{g^2}\,\mathrm{Tr}\left(\frac{1}{2}\left([\overline{x},x]+[\overline{y},y]\right)^2 + 2\Big|[x,y]\Big|^2 + \sqrt{2}\left(\alpha[\overline{x},\lambda]+\beta[\overline{y},\lambda]-\alpha[y,\xi]+\beta[x,\xi]\right)\right). \tag{2.16}$$

The location of each variable on the lattice can be read off from the corresponding $\mathbf{r}$ charges given in Table 1: the λ fermion lives at the sites, while ξ lives along the diagonal links; x, $\overline{x}$ and α reside along the $\hat{\mathbf{i}}$-links; y, $\overline{y}$ and β reside along the $\hat{\mathbf{j}}$-links. The lattice action is found by replacing all of the matrix variables in the action of the mother theory eq. (2.16) by their orbifold projection[5]

$$\begin{aligned} S = \frac{1}{g^2}\sum_{\mathbf{n}} \mathrm{Tr}\Bigg[&\frac{1}{2}\left(\overline{x}_{\mathbf{n}-\hat{\mathbf{i}}}x_{\mathbf{n}-\hat{\mathbf{i}}} - x_{\mathbf{n}}\overline{x}_{\mathbf{n}} + \overline{y}_{\mathbf{n}-\hat{\mathbf{j}}}y_{\mathbf{n}-\hat{\mathbf{j}}} - y_{\mathbf{n}}\overline{y}_{\mathbf{n}}\right)^2 + 2\left|x_{\mathbf{n}}y_{\mathbf{n}+\hat{\mathbf{i}}} - y_{\mathbf{n}}x_{\mathbf{n}+\hat{\mathbf{j}}}\right|^2 + \\ &+\sqrt{2}\left(\alpha_{\mathbf{n}}\overline{x}_{\mathbf{n}}\lambda_{\mathbf{n}} - \alpha_{\mathbf{n}-\hat{\mathbf{i}}}\lambda_{\mathbf{n}}\overline{x}_{\mathbf{n}-\hat{\mathbf{i}}}\right) + \sqrt{2}\left(\beta_{\mathbf{n}}\overline{y}_{\mathbf{n}}\lambda_{\mathbf{n}} - \beta_{\mathbf{n}-\hat{\mathbf{j}}}\lambda_{\mathbf{n}}\overline{y}_{\mathbf{n}-\hat{\mathbf{j}}}\right) - \\ &-\sqrt{2}\left(\alpha_{\mathbf{n}}y_{\mathbf{n}+\hat{\mathbf{i}}}\xi_{\mathbf{n}} - \alpha_{\mathbf{n}+\hat{\mathbf{j}}}\xi_{\mathbf{n}}y_{\mathbf{n}}\right) + \sqrt{2}\left(\beta_{\mathbf{n}}x_{\mathbf{n}+\hat{\mathbf{j}}}\xi_{\mathbf{n}} - \beta_{\mathbf{n}+\hat{\mathbf{i}}}\xi_{\mathbf{n}}x_{\mathbf{n}}\right)\Bigg], \end{aligned} \tag{2.17}$$

where the vectors $\mathbf{n} = \{n_x, n_y\}$ label the lattice sites on the $N \times N$ periodic lattice of Fig. 1.

3. Symmetries of the lattice action

The seemingly elaborate construction of the above lattice action is warranted by the symmetry properties it possesses — symmetries which ensure that the target theory arises in the continuum limit without fine tuning of operators. The symmetries of the lattice action

[5]We label the coordinates of our variables such that $\lambda_{\mathbf{n}}$ resides at site $\mathbf{n}$; $x_{\mathbf{n}}$, $\overline{x}_{\mathbf{n}}$ and $\alpha_{\mathbf{n}}$ live on the $(\mathbf{n}, \mathbf{n}+\hat{\mathbf{i}})$ link; $y_{\mathbf{n}}$, $\overline{y}_{\mathbf{n}}$ and $\beta_{\mathbf{n}}$ live on the $(\mathbf{n}, \mathbf{n}+\hat{\mathbf{j}})$ link; and $\xi_{\mathbf{n}}$ sits on the $(\mathbf{n}, \mathbf{n}+\hat{\mathbf{i}}+\hat{\mathbf{j}})$ diagonal links. Link variables have an orientation determined by their vector $\mathbf{r}$. Shown in figure 1 are the orientations of the link fermions, α, β and ξ. The bosonic link variables x and y are oriented in the same sense as α and β respectively, while $\overline{x}$ and $\overline{y}$ are anti-oriented.

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include all symmetries of the mother theory which commute with the orbifold projection operator $\hat{P}$, defined previously as

$$\hat{P} = \frac{1}{N^2} \sum_{m,n=1}^{N} (\hat{\gamma}_1)^m (\hat{\gamma}_2)^n \,, \qquad \hat{\gamma}_a \Phi = e^{2\pi i r_a/N} \mathcal{C}^{(a)} \Phi \mathcal{C}^{(a)^{-1}} \qquad \text{with} \quad a = 1, 2 \,. \tag{3.1}$$

These consist of a U(k) gauge symmetry (e.g. a $\mathrm{U}(k)^{N^2}$ symmetry, with an independent U(k) symmetry associated with each site); a $\mathbb{Z}_N \times \mathbb{Z}_N$ discrete translational symmetry; a $\mathrm{U}(1)^3$ global internal symmetry; a $\mathbb{Z}_2$ lattice point group consisting of reflections about the diagonal link; and a $\mathcal{Q} = 1$ supersymmetry. These symmetries are important for determining the renormalization properties of the theory, and so we demonstrate how each of these symmetries commutes with the orbifold projection and describe the action on component fields. We pay particular attention to the supersymmetry, and reexpress the lattice action eq. (2.17) in terms of superfields so that the possible form of counterterms becomes transparent.

3.1 The U(k) gauge symmetry

The U(k) gauge symmetry corresponds to an independent U(k) symmetry associated with each of the N^2 lattice sites. This $\mathrm{U}(k)^{N^2}$ symmetry is a subgroup of the $\mathcal{G} = \mathrm{U}(kN^2)$ symmetry of the mother theory. Using the direct product notation $\Phi_{\mu\nu i,\mu'\nu' j}$, where μ, ν, μ', ν' run from $1,\ldots,N$ and i, j run from $1,\ldots,k$, for any of the $kN^2 \times kN^2$ matrices of the mother theory, the action of the $\mathrm{U}(k)^{N^2}$ symmetry may be written as

$$\Phi \to \mathcal{U}\Phi\mathcal{U}^\dagger \,, \qquad \mathcal{U}_{\mu\nu i,\mu'\nu' j} = U^{(\mu\nu)}_{ij} \delta_{\mu\mu'} \delta_{\nu\nu'} \,, \tag{3.2}$$

where $U^{(\mu\nu)}$ is an independent rank-k unitary matrix for each of the N^2 values of (μ, ν). This transformation commutes with the orbifold projection eq. (3.1): it acts nontrivially only in the $k \times k$ subspace corresponding to indices i, j, while the clock matrices $\mathcal{C}^{(a)}$ act nontrivially only on the μ, ν indices

$$\mathcal{C}^{(1)}_{\mu\nu i,\mu'\nu' j} = \omega^\mu \delta_{\mu\mu'} \delta_{\nu\nu'} \delta_{ij} \,, \qquad \mathcal{C}^{(2)}_{\mu\nu i,\mu'\nu' j} = \omega^\nu \delta_{\mu\mu'} \delta_{\nu\nu'} \delta_{ij} \qquad \text{with} \quad \omega \equiv e^{2\pi i/N} \,. \tag{3.3}$$

This U(k) gauge symmetry also commutes with the r_a charges that appear in the projection operator $\hat{P}$, as they live in a different space.

A site variable on the lattice such as $\lambda_{\mathbf{n}}$ transforms as an adjoint under the $\mathrm{U}^{(\mathbf{n})}(k)$ symmetry associated with that site; link variables such as $x_{\mathbf{n}}$ or $\overline{y}_{\mathbf{n}}$ transform as bifundamentals under the two U(k) symmetries associated with the sites at the link's endpoints. The orientation of a link variable is determined by its $\mathbf{r}$ charges. For example, since $x_{\mathbf{n}}$ has $\mathbf{r} = \{1, 0\}$, it transforms as $x_{\mathbf{n}} \to \mathrm{U}^{(\mathbf{n})}\, x_{\mathbf{n}}\, \mathrm{U}^{(\mathbf{n}+\hat{\mathbf{i}})\,\dagger}$, while $\overline{y}_{\mathbf{n}}$ has $\mathbf{r} = (0, -1)$ and so transforms as $\overline{y}_{\mathbf{n}} \to \mathrm{U}^{(\mathbf{n}+\hat{\mathbf{j}})}\, \overline{y}_{\mathbf{n}}\, \mathrm{U}^{(\mathbf{n})\,\dagger}$. Gauge invariant operators are constructed from traces of products of variables, corresponding to oriented closed loops on the lattice.

3.2 The $\mathbb{Z}_N \times \mathbb{Z}_N$ translation symmetry

The translation symmetries of the lattice also arise from $\mathcal{G}$ symmetries of the mother theory. We define the "shift" matrices

$$\mathcal{T}^{(1)} = T \otimes \mathbf{1}_N \otimes \mathbf{1}_k \,, \qquad \mathcal{T}^{(2)} = \mathbf{1}_N \otimes T \otimes \mathbf{1}_k \,, \tag{3.4}$$

where

$$T = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}. \tag{3.5}$$

The two shift operators $\hat{t}_a$ are defined to act on the fields of the mother theory as $\hat{t}_a\Phi = \mathcal{T}^{(a)}\Phi\mathcal{T}^{(a)}{}^{-1}$, and they generate a $\mathbb{Z}_N \times \mathbb{Z}_N$ subgroup of $\mathcal{G}$. These shift matrices commute with the r_a charges, while conjugation of the clock matrices by the shift matrices is simply

$$\mathcal{T}^{(a)}\mathcal{C}^{(b)}\mathcal{T}^{(a)}{}^{-1} = \begin{cases} \omega\, \mathcal{C}^{(b)}\,, & a = b \\ \mathcal{C}^{(b)}\,, & a \neq b\,. \end{cases} \tag{3.6}$$

It follows that the shift operators $\hat{t}_a$ commute with the $\mathbb{Z}_N \times \mathbb{Z}_N$ generators $\hat{\gamma}_b$, and hence with the orbifold projection operator $\hat{P}$:

$$[\hat{t}_a, \hat{\gamma}_b] = [\hat{t}_a, \hat{P}] = 0 \qquad \text{with} \quad a, b = 1, 2\,. \tag{3.7}$$

Since the $\hat{t}_a$ commute with $\hat{P}$, they generate symmetries of the lattice action. By considering the action of the shift matrices $\mathcal{T}^{(a)}$ on the lattice component fields, one can see that the $\hat{t}_a$ generate the $\mathbb{Z}_N \times \mathbb{Z}_N$ lattice translations, with $\hat{t}_1$ generating translations in the $\hat{\mathbf{i}}$ direction, and $\hat{t}_2$ generating translations in the $\hat{\mathbf{j}}$ direction. Note that the $\hat{t}_a$ as defined do not commute with the U(k) gauge symmetry.

3.3 The U$(1)^3$ global symmetry

The U$(1)^3$ global symmetry is generated by r_1, r_2, and Y. These charges live in the Cartan sub-algebra of the G_R symmetry group of the mother theory, and so commute with the $\mathcal{C}^{(a)} \in \mathcal{G}$ in eq. (3.1). They also clearly commute with the $r_a \in G_R$, which we chose to equal $\pm L_3 + R_3 - Y$. Since gauge invariant operators correspond to closed loops on the lattice, and since r_1 and r_2 specify the link vectors, each gauge invariant operator will necessarily have net charges $r_1 = r_2 = 0$. The remaining U(1) generated by Y is nontrivial, however, and corresponds to an exact U(1) R-symmetry on the lattice, which in turn becomes the exact U(1) fermion number symmetry of the target theory in eq. (1.1).

3.4 The $\mathbb{Z}_2$ point group

The lattice in figure 1 respects a $\mathbb{Z}_2$ symmetry, corresponding to reflections about the diagonal link. The generator $\hat{\sigma}_d$ of this symmetry resides in both the G_R and the $\mathcal{G}$ groups of the mother theory. Its action on a field Φ of the mother theory is

$$\hat{\sigma}_d\, \Phi = \hat{g}\, \mathcal{D}\Phi\mathcal{D}^{-1} \qquad \text{with} \quad \hat{g} = e^{i\pi(L_1+R_3+Y)} \in G_R\,, \quad \mathcal{D}_{\mu\nu i,\mu'\nu' j} = \delta_{\mu\nu'}\delta_{\nu\mu'}\delta_{ij} \in \mathcal{G}\,. \tag{3.8}$$

These matrices have the properties

$$\hat{g}\, e^{2\pi i r_1/N}\hat{g}^{-1} = e^{2\pi i r_2/N} \qquad\qquad \mathcal{D}\,\mathcal{C}^{(1)}\mathcal{D}^{-1} = \mathcal{C}^{(2)} \tag{3.9a}$$

$$\hat{g}\, e^{2\pi i r_2/N}\hat{g}^{-1} = e^{2\pi i r_1/N} \qquad\qquad \mathcal{D}\,\mathcal{C}^{(2)}\mathcal{D}^{-1} = \mathcal{C}^{(1)}\,. \tag{3.9b}$$

This $\mathbb{Z}_2$ interchanges the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ axes of the daughter theory, by interchanging the two $\mathbb{Z}_N$ symmetries of the orbifold projection, $\hat{\sigma}_d\hat{\gamma}_1\hat{\sigma}_d^{-1} = \hat{\gamma}_2$, $\hat{\sigma}_d\hat{\gamma}_2\hat{\sigma}_d^{-1} = \hat{\gamma}_1$. It follows immediately that $\hat{\sigma}_d$ commutes with the orbifold projection operator $\hat{P}$ as defined in eq. (3.1). The action of $\hat{\sigma}_d$ on the lattice variables is

$$
\begin{aligned}
x_{\mathbf{n}} &\to y_{\tilde{\mathbf{n}}} & \alpha_{\mathbf{n}} &\to \beta_{\tilde{\mathbf{n}}} & \text{(3.10a)}\\
\overline{x}_{\mathbf{n}} &\to \overline{y}_{\tilde{\mathbf{n}}} & \beta_{\mathbf{n}} &\to \alpha_{\tilde{\mathbf{n}}} & \text{(3.10b)}\\
y_{\mathbf{n}} &\to x_{\tilde{\mathbf{n}}} & \lambda_{\mathbf{n}} &\to \lambda_{\tilde{\mathbf{n}}} & \text{(3.10c)}\\
\overline{y}_{\mathbf{n}} &\to \overline{x}_{\tilde{\mathbf{n}}} & \xi_{\mathbf{n}} &\to -\xi_{\tilde{\mathbf{n}}}\,, & \text{(3.10d)}
\end{aligned}
$$

where $\mathbf{n} = \{n_x, n_y\}$ and $\tilde{\mathbf{n}} = \{n_y, n_x\}$.

3.5 The $\mathcal{Q} = 1$ supersymmetry and a superfield formulation

Our method for constructing the lattice action for (2,2) SYM in two dimensions preserves an exact supersymmetry. Since the supersymmetry transformations of the mother theory in eq. (2.6) are gauge covariant, it follows that the four supercharges Q_i of the mother theory are gauge invariant. Therefore the $\hat{\gamma}_a$ generators of the Γ symmetry used in the orbifold projection act on the supercharges simply as G_R transformations:

$$\hat{\gamma}_a\, Q_i = \left(e^{2\pi i r_a/N}\right)_{ij} Q_j\,. \tag{3.11}$$

Any linear combination of these four supercharges which have $\mathbf{r} = \{0,0\}$ are invariant under $\mathbb{Z}_N \times \mathbb{Z}_N$ and survive the orbifold projection.

Consider the first of the transformations in eq. (2.6)

$$\delta v_m = -i\overline{\psi}\,\overline{\sigma}_m\kappa + i\overline{\kappa}\,\overline{\sigma}_m\psi\,. \tag{3.12}$$

This is G_R covariant if we consider the Grassmann spinor parameters κ and $\overline{\kappa}$ as transforming under G_R in the same manner as ψ and $\overline{\psi}$ respectively. It follows that the supercharges of the mother theory transform under the G_R representation that is conjugate to that of the gauginos ψ and $\overline{\psi}$, namely $(2,1)_{+1/2} \oplus (1,2)_{-1/2}$. Since one of the four components of ψ and $\overline{\psi}$ has $\mathbf{r} = \{0,0\}$ — namely the component λ — one of the four supercharges has charge $\mathbf{r} = \{0,0\}$ and survives the orbifold projection. This is a general result: for supersymmetric lattices produced by orbifold projection from an SYM matrix model, the number of unbroken supercharges equals the number of gaugino site variables on the lattice.

To analyze the action of the unbroken supercharge, we write the Grassmann spinors κ and $\overline{\kappa}$ which parameterize the $\mathcal{Q} = 4$ supersymmetry transformations of the mother theory in terms of the four independent components

$$\kappa = \begin{pmatrix}\eta\\ \eta'\end{pmatrix}, \qquad \overline{\kappa} = \begin{pmatrix}\overline{\eta} & \overline{\eta}'\end{pmatrix}. \tag{3.13}$$

With these definitions the upper component of κ, labeled as η in eq. (3.13), transforms the same way as the spinor λ under G_R and has $\mathbf{r} = \{0,0\}$. Therefore the supercharge that generates supersymmetric translations in η survives the orbifold projection and gives rise to an exact supersymmetry of the lattice action.

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In order to study the renormalization properties of this lattice theory, we rewrite the action in superfield notation to make the exact $\mathcal{Q} = 1$ supersymmetry more explicit. Setting $\eta' = \overline{\eta} = \overline{\eta}' = 0$ in eq. (3.13) and substituting into eq. (2.6), we find the supersymmetry transformations left unbroken by the orbifold projection:

$$\delta x = -\sqrt{2}\, i\, \alpha\, \eta \qquad \delta\overline{x} = 0 \tag{3.14a}$$
$$\delta y = -\sqrt{2}\, i\, \beta\, \eta \qquad \delta\overline{y} = 0 \tag{3.14b}$$
$$\delta\lambda = -i\left([\overline{x}, x] + [\overline{y}, y]\right)\eta \qquad \delta\alpha = 0 \tag{3.14c}$$
$$\delta\xi = 2i[\overline{x}, \overline{y}]\,\eta \qquad \delta\beta = 0\,. \tag{3.14d}$$

To construct a superfield notation that realizes the supersymmetry algebra off-shell, we define the supercharge Q in terms of a Grassmann coordinate θ:

$$\delta = i\eta Q\,, \qquad Q = \frac{\partial}{\partial\theta}\,. \tag{3.15}$$

The supersymmetry variations as written in eq. (3.14) are not consistent with the definition of $\delta = i\eta\partial_\theta$ as they stand, as $\delta^2\lambda \neq 0$. We remedy this by introducing an auxiliary field d such that

$$\begin{aligned} \delta\lambda &= -i\left([\overline{x}, x] + [\overline{y}, y] + id\right)\eta \\ \delta d &= \sqrt{2}\left([\overline{x}, \alpha] + [\overline{y}, \beta]\right)\eta \end{aligned} \tag{3.16}$$

which yields $\delta^2\lambda = 0$. Making use of the equations of motion we will find $d = 0$ and $\delta d = 0$.

With $\delta = i\eta\partial_\theta$, these transformation laws can be realized in terms of the following superfields:

$$\mathbf{X} = x + \sqrt{2}\,\theta\,\alpha \tag{3.17a}$$
$$\mathbf{Y} = y + \sqrt{2}\,\theta\,\beta \tag{3.17b}$$
$$\mathbf{\Lambda} = \lambda - \left([\overline{x}, x] + [\overline{y}, y] + id\right)\,\theta \tag{3.17c}$$
$$\mathbf{\Xi} = \xi + 2[\overline{x}, \overline{y}]\,\theta\,. \tag{3.17d}$$

The action of the mother theory can be written in terms of these superfields, as well as $\overline{x}$ and $\overline{y}$, which are supersymmetric singlets (that is, matrices with no θ dependence).

After orbifolding, we label each surviving $k \times k$ block of the original superfields by their lattice coordinate. The lattice variables are the θ-independent matrices $\overline{x}_{\mathbf{n}}$ and $\overline{y}_{\mathbf{n}}$, as well as the superfields

$$\mathbf{X}_{\mathbf{n}} = x_{\mathbf{n}} + \sqrt{2}\,\theta\,\alpha_{\mathbf{n}} \tag{3.18a}$$
$$\mathbf{Y}_{\mathbf{n}} = y_{\mathbf{n}} + \sqrt{2}\,\theta\,\beta_{\mathbf{n}} \tag{3.18b}$$
$$\mathbf{\Lambda}_{\mathbf{n}} = \lambda_{\mathbf{n}} - \left((\overline{x}_{\mathbf{n}-\hat{\mathbf{i}}}x_{\mathbf{n}-\hat{\mathbf{i}}} - x_{\mathbf{n}}\overline{x}_{\mathbf{n}}) + (\overline{y}_{\mathbf{n}-\hat{\mathbf{j}}}y_{\mathbf{n}-\hat{\mathbf{j}}} - y_{\mathbf{n}}\overline{y}_{\mathbf{n}}) + id_{\mathbf{n}}\right)\,\theta \tag{3.18c}$$
$$\mathbf{\Xi}_{\mathbf{n}} = \xi_{\mathbf{n}} + 2\left(\overline{x}_{\mathbf{n}+\hat{\mathbf{j}}}\overline{y}_{\mathbf{n}} - \overline{y}_{\mathbf{n}+\hat{\mathbf{i}}}\overline{x}_{\mathbf{n}}\right)\,\theta\,. \tag{3.18d}$$

These superfields are all $k \times k$ matrices. Note that the exact lattice supersymmetry allows for variables which reside at different places on the lattice to exist in the same super multiplet. For example, $\mathbf{\Xi_n}$ depends on $\xi_{\mathbf{n}}$, which sits on the diagonal link, as well as the $\bar{x}$ and $\bar{y}$ link variables residing at the edges of the plaquette that the $\xi_{\mathbf{n}}$ link bisects. In fact, the θ term of $\mathbf{\Xi}$ resembles the square root of the usual plaquette term found in lattice QCD. It is this slightly nonlocal structure on the lattice that allows successive supersymmetry transformations to generate translations in the continuum limit.

In terms of these fields, the lattice action eq. (2.17) may be written in a manifestly $\mathcal{Q} = 1$ supersymmetric form:

$$
\begin{aligned}
S = \frac{1}{g^2}\int d\theta \sum_{\mathbf{n}} \mathrm{Tr}\Bigg[-\frac{1}{2}\mathbf{\Lambda_n}\,\partial_\theta \mathbf{\Lambda_n} - & \\
& - \mathbf{\Lambda_n}(\bar{x}_{\mathbf{n}-\hat{\mathbf{i}}}\mathbf{X}_{\mathbf{n}-\hat{\mathbf{i}}} - \mathbf{X_n}\bar{x}_{\mathbf{n}} + \bar{y}_{\mathbf{n}-\hat{\mathbf{j}}}\mathbf{Y}_{\mathbf{n}-\hat{\mathbf{j}}} - \mathbf{Y_n}\bar{y}_{\mathbf{n}}) - \\
& - \mathbf{\Xi_n}\,(\mathbf{X_n}\mathbf{Y}_{\mathbf{n}+\hat{\mathbf{i}}} - \mathbf{Y_n}\mathbf{X}_{\mathbf{n}+\hat{\mathbf{j}}})\Bigg]\,.
\end{aligned}
\tag{3.19}
$$

The lattice action eq. (2.17) may be recovered by integrating over the auxiliary field $d_{\mathbf{n}}$. Note that $d_{\mathbf{n}}$ is a decoupled gaussian variable — its only function is to allow closure of the supersymmetry algebra.

This action has a U(1) R-symmetry under which θ, λ and ξ have charge $+\frac{1}{2}$, α and β have charge $-\frac{1}{2}$, while x, $\bar{x}$, y, and $\bar{y}$ have charge zero. This is just the $\mathrm{U}(1)_Y$ symmetry discussed in § 3.3. In the continuum this becomes the vector U(1) symmetry of the target theory eq. (1.1). Under the $\mathbb{Z}_2$ crystal symmetry eq. (3.10), the superfields transform as

$$
\begin{aligned}
&\mathbf{\Lambda_n} \to \mathbf{\Lambda_{\tilde{n}}} && \\
&\mathbf{X_n} \to \mathbf{Y_{\tilde{n}}} && \bar{x}_{\mathbf{n}} \to \bar{y}_{\tilde{\mathbf{n}}} \\
&\mathbf{Y_n} \to \mathbf{X_{\tilde{n}}} && \bar{y}_{\mathbf{n}} \to \bar{x}_{\tilde{\mathbf{n}}} \\
&\mathbf{\Xi_n} \to -\mathbf{\Xi_{\tilde{n}}}\,. &&
\end{aligned}
\tag{3.20}
$$

4. The continuum limit: tree level

The action eq. (2.17) has the "topology" of a lattice action, but as yet no dimensionful parameter identifiable as a lattice spacing. However, the action has a moduli space: flat directions in the bosonic variables. For example, the potential vanishes for translationally invariant configurations of the form $x = \bar{x}$, $y = \bar{y}$ and $[x, y] = 0$. We consider the particular configuration which preserves the $\mathbb{Z}_2$ lattice symmetry and a $\mathrm{U}(k)$ global symmetry, namely

$$x_{\mathbf{n}} = \bar{x}_{\mathbf{n}} = y_{\mathbf{n}} = \bar{y}_{\mathbf{n}} = \frac{1}{a\sqrt{2}}\mathbf{1}_k\,, \tag{4.1}$$

where a is a parameter with dimensions of length, and $\mathbf{1}_k$ is the $k\times k$ unit matrix. We expand the action about this classical configuration eq. (4.1), identifying the lattice spacing with a. Then the continuum limit is taken as

$$a \to 0\,, \qquad N \to \infty\,, \qquad g^2 \to \infty\,, \qquad g^2a^2 = g_2^2\,, \qquad aN = L\,, \tag{4.2}$$

where the two-dimensional gauge coupling g_2 and the box size L are kept fixed. The infinite volume limit, $L \to \infty$ may be taken separately. Because the target theory is superrenormalizable, the dimensionless constant $g_2^2 a^2$ vanishes in the continuum limit, justifying a perturbative discussion of the continuum limit of this model. All nonperturbative phenomena will be associated with infrared physics.

To analyze the lattice eq. (2.17) we write x and y in terms of their hermitean and anti-hermitean parts:

$$x = \overline{x}^\dagger = \frac{1}{a\sqrt{2}} + \frac{s_1 + iv_1}{\sqrt{2}}\,, \qquad y = \overline{y}^\dagger = \frac{1}{a\sqrt{2}} + \frac{s_2 + iv_2}{\sqrt{2}}\,, \tag{4.3}$$

where both s_i and v_i are hermitean $k \times k$ matrices. Expanding about the point in moduli space eq. (4.1) and taking the limit eq. (4.2), we show below that the lattice action eq. (2.17) becomes the action of the target theory eq. (1.1) where the continuum variables are defined in terms of the lattice variables as

$$s = \frac{s_1 + is_2}{\sqrt{2}}\,, \qquad v_m = \{v_1, v_2\}\,, \qquad \psi = \begin{pmatrix} \lambda \\ \xi \end{pmatrix}\,, \qquad \overline{\psi} = i\begin{pmatrix} \alpha & \beta \end{pmatrix}\,, \tag{4.4}$$

in the Dirac γ-matrix basis

$$\gamma_1 = \sigma_3\,, \quad \gamma_2 = \sigma_1\,, \quad \gamma_3 = \sigma_2\,. \tag{4.5}$$

The matrix γ_3 is the chirality matrix in two dimensions, with $\psi_{R,L}$ in eq. (1.1) defined as $\frac{1\pm\gamma_3}{2}\,\psi$.

While simple to derive, the above result is startling. For one thing, the lattice has no discrete rotational symmetry — the $\mathbb{Z}_2$ crystal symmetry behaves more like parity — nevertheless an SO(2) euclidean rotation symmetry is recovered in the continuum. Furthermore, the $s_{1,2}$ variables live on the links, leading to the natural assumption that they transform nontrivially under spacetime rotations in the continuum limit. Instead $s = (s_1 + is_2)/\sqrt{2}$ is a Lorentz singlet in the continuum; the SO(2) symmetry transforming s_1 and s_2 into each other ends up being part of the internal chiral $\mathrm{U}(1)_R$ symmetry of the target theory, and not a spacetime symmetry. In contrast, the anti-hermitean parts of the link variables $v_{1,2}$ indeed transform nontrivially under Lorentz rotations. Finally, the four one-component fermions, each located at different points on the lattice, arrange themselves into a single Dirac spinor in the continuum limit, with no extraneous doublers.

To determine the continuum limit of our lattice action, we begin by considering the free kinetic terms at the classical level. After eliminating the auxiliary field d from the lattice action eq. (3.19) and expanding about the point in moduli space given in eq. (4.1), the terms quadratic in the bosonic variables s and v are

$$\frac{1}{2g^2a^2}\sum_{\mathbf{n}}\mathrm{Tr}\Bigg[\left(s_{1,\mathbf{n}-\hat{\mathbf{i}}}-s_{1,\mathbf{n}}+s_{2,\mathbf{n}-\hat{\mathbf{j}}}-s_{2,\mathbf{n}}\right)^2+$$
$$+\left|\left(s_{1,\mathbf{n}+\hat{\mathbf{j}}}-s_{1,\mathbf{n}}+s_{2,\mathbf{n}}-s_{2,\mathbf{n}+\hat{\mathbf{i}}}\right)-i\left(v_{1,\mathbf{n}+\hat{\mathbf{j}}}-v_{1,\mathbf{n}}-v_{2,\mathbf{n}+\hat{\mathbf{i}}}+v_{2,\mathbf{n}}\right)\right|^2\Bigg]=$$
$$=\frac{1}{2g^2}\sum_{\mathbf{n}}\mathrm{Tr}\left[\sum_{\hat{\boldsymbol{\mu}}}\sum_{i=1,2}\left(\frac{s_{i,\mathbf{n}}-s_{i,\mathbf{n}-\hat{\boldsymbol{\mu}}}}{a}\right)^2+\left(\frac{v_{1,\mathbf{n}+\hat{\mathbf{j}}}-v_{1,\mathbf{n}}}{a}-\frac{v_{2,\mathbf{n}+\hat{\mathbf{i}}}-v_{2,\mathbf{n}}}{a}\right)^2\right], \quad (4.6)$$

where $\hat{\boldsymbol{\mu}}$ is summed over $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$. We can diagonalize these kinetic terms by introducing the Fourier transform of a field ϕ

$$\phi_{\mathbf{n}}\equiv\frac{1}{N}\sum_{\mathbf{p}}e^{ia\mathbf{p}\cdot\mathbf{n}}\phi_{\mathbf{p}}\,, \quad (4.7)$$

where the momenta p_i are integers times π/L, lying in the Brillouin zone $p_i\in[-\frac{\pi}{a},\frac{\pi}{a})$. The bosonic "hopping" terms eq. (4.6) become

$$\frac{1}{2g^2}\sum_{\mathbf{p}}\mathrm{Tr}\left[\left(\mathcal{P}_x^2+\mathcal{P}_y^2\right)\left(s_{1,\mathbf{p}}s_{1,-\mathbf{p}}+s_{2,\mathbf{p}}s_{2,-\mathbf{p}}\right)+\begin{pmatrix}v_{1,\mathbf{p}} & v_{2,\mathbf{p}}\end{pmatrix}G(\mathbf{p})\begin{pmatrix}v_{1,-\mathbf{p}}\\ v_{2,-\mathbf{p}}\end{pmatrix}\right], \quad (4.8)$$

where we have defined

$$\mathcal{P}_i\equiv\frac{2}{a}\sin\frac{ap_i}{2}\,,\qquad \mathbf{G}(\mathbf{p})\equiv\begin{pmatrix}\mathcal{P}_y^2 & -e^{-ia(p_x-p_y)/2}\mathcal{P}_x\mathcal{P}_y\\ -e^{ia(p_x-p_y)/2}\mathcal{P}_x\mathcal{P}_y & \mathcal{P}_x^2\,.\end{pmatrix} \quad (4.9)$$

The eigenvalues of $\mathbf{G}$ are $(\mathcal{P}_x^2+\mathcal{P}_y^2)$ and zero. In the continuum limit $\mathcal{P}_i\to p_i$ and $\mathbf{G}(\mathbf{p})_{ij}\to(\mathbf{p}^2\delta_{ij}-p_ip_j)$. We see that the continuum limit of the boson lattice action correctly reproduces the kinetic terms of the target theory

$$\frac{1}{2g_2^2}\int d^2z\ \mathrm{Tr}\left[\boldsymbol{\nabla}s^\dagger\cdot\boldsymbol{\nabla}s+(\partial_xv_2-\partial_yv_1)^2\right], \quad (4.10)$$

where we used the definitions eq. (4.4) and the substitution $a^2\sum_{\mathbf{n}}\to\int d^2z$ as $a\to 0$. Note that the above lagrangean has both a rotational symmetry under which s is a scalar and $\mathbf{v}$ is a vector, as well as a U(1) symmetry under which s is charged and v is neutral — neither symmetry being apparent in the lattice action.

Following the same procedure for the fermionic part of the lattice action eq. (2.17), we find the noninteracting part of the fermion hopping terms

$$\frac{1}{g^2}\sum_{\mathbf{n}}\mathrm{Tr}\begin{pmatrix}\alpha_{\mathbf{p}} & \beta_{\mathbf{p}}\end{pmatrix}i\mathbf{K}(\mathbf{p})\begin{pmatrix}\lambda_{-\mathbf{p}}\\ \xi_{-\mathbf{p}}\end{pmatrix} \quad (4.11)$$

with the definition

$$\mathbf{K}(\mathbf{p})=\begin{pmatrix}e^{-iap_x/2}\mathcal{P}_x & e^{iap_y/2}\mathcal{P}_y\\ e^{-iap_y/2}\mathcal{P}_y & -e^{iap_x/2}\mathcal{P}_x\end{pmatrix}. \quad (4.12)$$

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The matrix $\mathbf{K}^\dagger\mathbf{K}$ is just the identity matrix times $(\mathcal{P}_x^2 + \mathcal{P}_y^2)$, which equals the scalar hopping matrix, as expected on a supersymmetric lattice. Note that there is no fermion doubling at the edges of the Brillouin zone, $ap_i = \pm\pi$. and the continuum limit of the free part of the fermion action is just the two dimensional Dirac action

$$\frac{1}{g_2^2}\int d^2z \; \mathrm{Tr}\, \overline{\psi}\, i\boldsymbol{\nabla}\cdot\boldsymbol{\gamma}\,\psi\,. \tag{4.13}$$

with ψ as defined in eq. (4.4), and γ matrices defined in eq. (4.5). This has the same form as in the target theory.

The above analysis has shown that only smooth configurations (up to gauge transformations) have small action, and we are justified in studying the continuum limit of the interacting theory by expanding about smooth field configurations, limiting ourselves to smooth gauge transformations. To retain manifest $\mathcal{Q} = 1$ supersymmetry, we define continuum versions of the lattice superfields in eq. (3.17); these are functions of $\mathbf{z}$, the spacetime coordinate in two euclidean dimensions:

$$\begin{aligned}
-\frac{i}{\sqrt{2}}(\mathbf{X_n} - \overline{x}_\mathbf{n}) \;&\rightarrow\; \mathbf{V}_1(\mathbf{z}) = v_1(\mathbf{z}) - i\theta\,\alpha(\mathbf{z})\\
-\frac{i}{\sqrt{2}}(\mathbf{Y_n} - \overline{y}_\mathbf{n}) \;&\rightarrow\; \mathbf{V}_2(\mathbf{z}) = v_2(\mathbf{z}) - i\theta\,\beta(\mathbf{z})\\
\frac{1}{\sqrt{2}}(\mathbf{X_n} + \overline{x}_\mathbf{n}) - \frac{1}{a} \;&\rightarrow\; \mathbf{S}_1(\mathbf{z}) = s_1(\mathbf{z}) + \theta\alpha(\mathbf{z})\\
\frac{1}{\sqrt{2}}(\mathbf{Y_n} + \overline{y}_\mathbf{n}) - \frac{1}{a} \;&\rightarrow\; \mathbf{S}_2(\mathbf{z}) = s_2(\mathbf{z}) + \theta\beta(\mathbf{z})\\
\boldsymbol{\Lambda}_\mathbf{n} \;&\rightarrow\; \boldsymbol{\Lambda}(\mathbf{z}) = \lambda(\mathbf{z}) + \theta\left(D_1 s_1(\mathbf{z}) + D_2 s_2(\mathbf{z}) - id(\mathbf{z}) + O(a)\right)\\
\boldsymbol{\Xi}_\mathbf{n} \;&\rightarrow\; \boldsymbol{\Xi}(\mathbf{z}) = \xi(\mathbf{z}) + \theta\left(D_2 s_1(\mathbf{z}) - D_1 s_2(\mathbf{z}) + [s_1, s_2] + iv_{12}(\mathbf{z}) + O(a)\right),
\end{aligned} \tag{4.14}$$

where $D_{1,2}$ are the gauge covariant derivatives, $D_m = \partial_m + i[v_m,\,\cdot\,]$, and v_{mn} is the gauge field strength. Note that the Grassmann superfields $\boldsymbol{\Lambda}$ and $\boldsymbol{\Xi}$ have $O(a)$ corrections.

Consider how the continuum fields transform under the lattice symmetries. Under smooth gauge transformations, all of these superfields transform homogeneously as $\Phi \rightarrow U\Phi U^\dagger$, with the exception of the gauge field multiplet which transforms in the usual way as $\mathbf{V}_m \rightarrow U\mathbf{V}_m U^\dagger + i(\partial_m U)U^\dagger$. By specifying "smooth" gauge transformations, we mean that the transformation laws have $O(a)$ corrections, which may be computed from eq. (4.14). The unbroken global U(1) symmetry — the $\mathrm{U}(1)_Y$ symmetry of the mother theory — acts as a $\mathrm{U}(1)_R$ symmetry on these fields, with $\boldsymbol{\Lambda}$, $\boldsymbol{\Xi}$ and the Grassmann coordinate θ carrying charge $-\frac{1}{2}$, while $\mathbf{V}_m$ and $\mathbf{S}_i$ have charge zero. Finally, the $\mathbb{Z}_2$ crystal symmetry interchanges the fields $\mathbf{V}_1 \leftrightarrow \mathbf{V}_2$ and $\mathbf{S}_1 \leftrightarrow \mathbf{S}_2$, while $\boldsymbol{\Xi} \rightarrow -\boldsymbol{\Xi}$ and $\boldsymbol{\Lambda} \rightarrow \boldsymbol{\Lambda}$; the coordinate also changes under the $\mathbb{Z}_2$, with $\mathbf{z} \rightarrow \tilde{\mathbf{z}}$, where $\mathbf{z} = \{x_1, x_2\}$ and $\tilde{\mathbf{z}} = \{x_2, x_1\}$.

In terms of these continuum $\mathcal{Q} = 1$ superfields, the lattice action eq. (3.19) has the continuum expansion

$$S = \frac{1}{g_2^2} \int d\theta \int d^2z \; \mathrm{Tr} \Bigg[-\frac{1}{2} \mathbf{\Lambda} \partial_\theta \mathbf{\Lambda} + \mathbf{\Lambda} \left(\mathcal{D}_1 \mathbf{S}_1 + \mathcal{D}_2 \mathbf{S}_2 \right) - \\ - \frac{1}{2} \mathbf{\Xi} \big(\mathcal{D}_1 \mathbf{S}_2 - \mathcal{D}_2 \mathbf{S}_1 + [\mathbf{S}_1 , \, \mathbf{S}_2] + i \mathbf{V}_{12} \big) \Bigg] + O(a) \,, \tag{4.15}$$

where we have defined a supersymmetric covariant derivative and field strength,

$$\mathcal{D}_m = \partial_m + i \big[\mathbf{V}_m , \, \cdot \, \big] \,, \qquad \mathbf{V}_{mn} = -i \, [\mathcal{D}_m , \, \mathcal{D}_n] \,. \tag{4.16}$$

It is not difficult to verify in component form that the above action agrees with the action of the target theory eq. (1.1).

The analysis up to this point has been purely classical. We now discuss renormalization and show that radiative effects do not spoil the nice features we found at tree level.

5. The continuum limit: renormalization

We turn now to the quantum theory. At tree level we have the desired continuum limit, but radiative effects could in principle induce unwanted relevant or marginal operators which do not respect the symmetries of the target theory. Following the analysis in ref. [22] we consider the lattice action expanded about the classical continuum limit in powers of the lattice spacing a. We then consider the possible operators that might be required as counterterms in a loop expansion, according to standard power counting arguments. Since the two-dimensional gauge coupling constant g_2 has mass dimension 1, the loop expansion is governed by powers of $g_2^2 a^2$, which vanishes in the continuum limit eq. (4.2), justifying a perturbative analysis. The allowed counterterms are restricted by the exact symmetries of the lattice theory. We show that the only possible counterterm allowed by these symmetries is a shift in the vacuum energy, which does not affect the physics of the target theory in the continuum limit.

Consider the radiative addition to the action of an operator $\mathcal{O}$

$$\delta S = \frac{1}{g_2^2} \int d\theta \int d^2z \; C_{\mathcal{O}} \, \mathcal{O} \,. \tag{5.1}$$

$\mathcal{O}$ must be Grassmann with $\mathrm{U}(1)_R$ charge equal to $-\frac{1}{2}$, and is constructed from the superfields, their θ-derivatives and covariant derivatives. For power counting purposes, $\int d^2z$ has dimension -2, θ has dimension $-\frac{1}{2}$, and $\int d\theta$ and ∂_θ scale with dimension $+\frac{1}{2}$. A contribution to $C_{\mathcal{O}}$ at ℓ loops is proportional to $g_2^{2\ell}$. Since the coupling g_2 has mass dimension 1, the loop expansion is an expansion in the dimensionless parameter $(g_2^2 a^2)^\ell$. Consequently the coefficient $C_{\mathcal{O}}$ of the operator $\mathcal{O}$ scaling with mass dimension p has a loop expansion

$$C_{\mathcal{O}} = a^{p-7/2} \sum_\ell c_\ell \, (g_2^2 a^2)^\ell \,, \tag{5.2}$$

where the dimensionless expansion coefficients c_ℓ depend at most logarithmically on the lattice spacing a.

The only possible local counterterms (operators whose coefficients do not vanish in the $a \to 0$ limit) are those for which $(p + 2\ell - 7/2) \le 0$. We have shown that at tree level $(\ell = 0)$ the effective action of the lattice theory agrees with that of the target theory, and so we need only consider $\ell \ge 1$, and hence $0 \le p \le 3/2$.

The superfield dimensions may be determined by the natural scaling of the lowest component, so the bosonic superfields $\mathbf{S}$ and $\mathbf{V}$ have dimension 1, while the Grassmann superfields $\mathbf{\Lambda}$ and $\mathbf{\Xi}$ scale with mass dimension $\frac{3}{2}$. Since the operator $\mathcal{O}$ must be Grassmann, our power counting arguments imply it must have the form $\mathrm{Tr}\, \partial_\theta \mathbf{B}$, where $\mathbf{B}$ is a bosonic superfield, or $\mathrm{Tr}\, \mathbf{\Gamma}$, where $\mathbf{\Gamma}$ is a Grassmann superfield. The requirement that $\mathcal{O}$ carry $\mathrm{U}(1)_R$ charge $-\frac{1}{2}$ precludes the former possibility, while the $\mathbb{Z}_2$ crystal symmetry forbids $\mathcal{O} = \mathrm{Tr}\, \mathbf{\Xi}$. The only possible counterterm is therefore

$$\frac{1}{g_2^2} \int d\theta \int d^2z \; i\xi \; \mathrm{Tr}\, \mathbf{\Lambda} = \frac{1}{g_2^2} \int d^2z \; \xi \; \mathrm{Tr}\, d \,, \tag{5.3}$$

where we have dropped integrals of total derivatives. This is a Fayet-Illiopoulos term for the $\mathrm{U}(1)$ part of the $\mathrm{U}(k)$ gauge group.

How does this term affect the theory? Since d enters the action as a free, gaussian field (recall that the d equation of motion is $d = 0$), this term shifts the value of d to a nonzero constant, which in turn merely adds a constant to the vacuum energy. After integrating d out of the theory, all that is left is a cosmological constant which has no affect on the spectrum nor on the interactions of the continuum limit of the theory.

We conclude that the target theory eq. (1.1) is obtained from the continuum limit of our lattice action without any fine tuning, modulo a possible uninteresting contribution to the vacuum energy density.

6. Moduli

The expansion of the lattice fields about the point

$$x_{\mathbf{n}} = \bar{x}_{\mathbf{n}} = y_{\mathbf{n}} = \bar{y}_{\mathbf{n}} = \frac{1}{a\sqrt{2}} \mathbf{1}_k \,, \tag{6.1}$$

in the classical moduli space of the lattice theory requires justification. In particular, quantum fluctuations about this vacuum must be small compared to the classical value. Before considering the lattice theory it is necessary to discuss the moduli of the continuum target theory. The action eq. (1.1) possesses a noncompact classical moduli space consisting of all constant scalar field configurations s satisfying $[s, s^\dagger] = 0$. The existence of these exact noncompact bosonic zeromodes in the target theory poses an apparent problem for the lattice theory. The integrals over these modes are formally divergent, and the expansion about eq. (6.1) is then poorly defined.

To eliminate the noncompact modes associated with any diagonal matrix value for s, up to gauge transformations, we add to the target theory a small mass term $\mu^2\, \mathrm{Tr}\, |s|^2$, which lifts the degeneracy of the moduli and favors $s = 0$. This term breaks supersymmetry softly (but not the gauge symmetry), and will be sent to zero as $1/L$ in the infinite volume limit. Physical questions on length scales smaller than $1/\mu$ (and hence smaller than the lattice size $1/L$) are unaffected by this mass term.

Eliminating the zeromodes from the target theory might seem insufficient — after all, the spontaneous breaking of compact symmetries cannot occur in two dimensions even if the zero mode of the order parameter is eliminated, due to the infrared divergences of the momentum $k \neq 0$ modes of the Goldstone bosons [36, 37]. Such is not the case here. Consider the correlator of the scalar field s in the continuum:

$$G(\mathbf{z_1}, \mathbf{z_2}) = \left| \langle s^\dagger(\mathbf{z_1}) s(\mathbf{z_2}) \rangle \right| . \tag{6.2}$$

Assuming that the mass term we add for s satisies $\mu \lesssim 1/L$, the contribution to $G(\mathbf{z_1}, \mathbf{z_2})$ from nonzero momentum modes is given by

$$G(\mathbf{z_1}, \mathbf{z_2}) \Big|_{k \neq 0} \sim g_2^2 \ln \frac{|\mathbf{z_1} - \mathbf{z_2}|}{L} \lesssim g_2^2 \ln \frac{a}{L} , \tag{6.3}$$

where we made explicit that the target theory is regulated in the ultraviolet by the lattice spacing a. We see that the root mean square fluctuations of s diverges in the infinite volume limit. This is a physical effect that cannot be eliminated from the theory. However, these infrared divergent fluctuations need not invalidate the expansion of the link variables about the classical values eq. (6.1), since the point we are expanding about corresponds to the ultraviolet cutoff, and not a physical scale, as would be the case for spontaneous symmetry breaking in two dimensions. So long as we take the continuum limit of the lattice theory such that

$$a^2 g_2^2 \ln L/a \to 0 , \tag{6.4}$$

the fluctuations of the modulus in the continuum theory do not destroy the lattice interpretation of our orbifold construction.

Now that we understand the moduli of the continuum theory, all that remains is to specify the modifications of the lattice action which implement these changes in the continuum. We introduce a small supersymmetry violating term in the lattice action to stabilize the noncompact bosonic zeromodes

$$\delta S = -\frac{1}{g^2} \int d\theta \sum_{\mathbf{n}} \mathrm{Tr} \left[a^2 \, \mathbf{F} \left[\left(\mathbf{X_n} \overline{x}_\mathbf{n} - \frac{1}{2a^2} \right)^2 + \left(\mathbf{Y_n} \overline{y}_\mathbf{n} - \frac{1}{2a^2} \right)^2 \right] \right] , \tag{6.5}$$

where $\mathbf{F}$ is a supersymmetry breaking spurion of the form

$$\mathbf{F} = \theta \mu^2 \tag{6.6}$$

and μ^2 is positive and proportional to the unit matrix. The addition of this term does not alter the UV properties of the theory. Expanding this new term about the continuum fields of eq. (4.14) leads to an addition to the action

$$\delta S = -\frac{1}{g_2^2} \int d\theta \int d^2 z \; \mathrm{Tr} \, \mathbf{F} \left(\mathbf{S}_1^2 + \mathbf{S}_2^2 \right) + O(a) = -\frac{1}{g_2^2} \int d^2 z \, \mu^2 \, \mathrm{Tr} \, |s|^2 + O(a) . \tag{6.7}$$

Classically this new term lifts the degeneracy of the vacua, giving the would-be moduli a positive mass squared μ^2 about the classical value $x = y = 1/a\sqrt{2}$, or equivalently, $s = 0$. As discussed above, we can take $\mu \sim 1/L = 1/Na$, and this mass term alters the vacuum of the continuum theory in an irrelevant way for correlators insensitive to mass scales of order $1/L$.

7. Discussion

We have shown how to construct a supersymmetric euclidean spacetime lattice. The present example has as its target theory $(2, 2)$ supersymmetry in two dimensions, but as was the case in ref. [22], the method is generalizable to theories with more supersymmetry and in higher dimensions. A detailed discussion of lattice theories possessing continuum limits with eight or sixteen conserved supercharges is in preparation. One could also try to generalize the four supercharge theory discussed here to include matter fields in the fundamental representation of the gauge group.

The present example does not require fine tuning, due to the underlying symmetries of the lattice. This is remarkable not only because the target theory is supersymmetric, but also because it possesses a chiral symmetry which is conserved up to anomalies. Nevertheless, practical obstacles will have to be overcome if the theory is to be simulated. In general, a dynamical simulation with light fermions (of mass $\lesssim 1/L$, where L is the lattice size) is required in order to obtain a SYM theory in the continuum limit. In addition, even though the target theory has a positive fermion determinant in the present example, the lattice theory presented here may not have. An interesting path to pursue which we have not explored would be to construct the Nicolai map [38, 39] for our lattice theory — this could in principle circumvent the usual problems with simulating dynamical fermions.[6]

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